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Linear and Area Measurement in Prekindergarten to Grade 2

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RESULTS from the NAEP international assessments indicate that students' understanding of measurement lags behind all other mathematics topics (National Center for Education Statistics 1996). In addition, several research reports reveal that even college-level students have difficulty with certain measurement concepts, in particular area measurement (cf., Baturo and Nason 1996; Simon and Blume 1994). "Something is clearly wrong with [measurement] instruction" (Kamii and Clark 1997) because it tends to focus on the procedures of measuring rather than the concepts underlying them. The reason for this procedural focus probably lies in the facts that measuring is a highly physical activity and that assessing the conceptual or mental understanding that accompanies such physical motion is not as clear to teachers as assessing students' skills. As a consequence, the goal of most instruction is to teach students how to measure, for example, with a conventional ruler. Most researchers agree, however, that the marks on a ruler and the procedures for measuring can mask the very conceptual activity that underlie the tool and the physical activity. Therefore, this article will examine the conceptual underpinnings of measuring rather than measuring skills. To this end, we explore length and area measurement and document both the conceptual building blocks of measuring and accompanying instructional ideas.

LENGTH MEASUREMENT

Length is a characteristic of an object and can be found by quantifying how far it is between the endpoints of the object. Distance refers to the empty space between two points. Measuring consists of two aspects: (1) identifying a unit of measure and *subdividing* (mentally and physically) the object by that unit and (2) placing that unit end to end (*iterating*) along-side the object being measured. The hash marks and numerals on a ruler, therefore, represent the result of iterating 12 inch-sized units.

Most researchers understand subdividing and unit iteration to be complex mental accomplishments that are largely downplayed in typical measurement teaching. As a consequence, rather than focus on only the physical act of measuring, much of the literature investigates students' understandings of measuring as *space covering.* (This is true for area as well; the space is one-dimensional for length and two-dimensional for area.) Students' early measuring experiences generally arise from counting: they count the number of times they iterate a unit. However, measuring is more complex than students' first counting experiences because the "objects" students count when measuring are continuous units (e.g., the length of a rug) rather than discrete units (e.g., fingers, blocks).

Important Concepts in Linear Measurement

There are several important concepts, or big ideas, that underpin much of learning to measure. It is important to understand these concepts so we can use them to understand how students are thinking about space as they go through the physical activity of measuring. These concepts are: (1) partitioning, (2) unit iteration, (3) transitivity, (4) conservation, (5) accumulation of distance, and (6) relation to number.

> **Partitioning** is the mental activity of slicing up the length of an object into the same-sized units.

The idea of partitioning a unit into smaller pieces is nontrivial for students and involves mentally seeing the length of the object as something that can be partitioned (or "cut up") before even physically measuring. Lehrer (in press) suggests that asking students to make their own rulers can reveal how they understand partitioning length. Some students, for instance, may draw hashmarks at uneven intervals, which indicates that they do not partition space into equal-sized units. As students come to understand that units are partitionable, they come to grips with the idea that length is continuous (e.g., any unit can itself be further partitioned).

> **Unit iteration** is the ability to think of the length of a small block as part of the length of the object being measured and to place the smaller block repeatedly along the length of the larger object (Kamii and Clark 1997).

Lehrer (in press) found that initially students may iterate a unit leaving gaps between subsequent units or overlapping adjacent units. For these students, iterating is a physical activity of placing units end-to-end in some manner, not an activity of covering the space or length of the object without gaps. Lehrer goes on to say that when students count each unit iteration, teachers should focus students' conversations on what each number word refers to. For example, if a student iterates a unit five times, the "five" represents five units of space. For some students "five" signifies the *hash mark* next to the numeral five instead of the amount of *space* covered by five units (see also Stephan et al. 2001). Additionally, many students see no problem mixing units (e.g., using both paper clips and pen tops) or using different-sized units (e.g., small and large paper clips) as long as they covered the entire length of the object in some way (Clements, Battista, and Sarama 1998; Lehrer in press). Furthermore, many studies, as well as national assessments, report that students begin counting at the numeral one on a ruler (i.e., 1 as the zero point; Lehrer in press) or, when counting paces heel-to-toe, start their count with the movement of the first foot (i.e., they miss the first foot and count the "second" foot as one from an adult perspective; Lehrer in press; Stephan et al. 2001). The researchers' explanation for this is that these students are not thinking about measuring as covering space. Rather, the numerals on a ruler (or the placement of a foot) signify when to start counting, not an amount of space that has already been covered (i.e., "one" is the space from the beginning of the ruler to the hash mark, not the hash mark itself). In this way, the marks on a ruler "mask" the intended conceptual understanding involved in measurement. A final issue related to unit iteration comes from Stephan et al. (2001) who found that many students initially find it necessary to iterate a unit until it "fills up" the length of the object and will not extend the unit past the endpoint of the object.

Transitivity is the understanding that:

(*a*) if the length of object 1 is equal to the length of object 2 and object 2 is the same length as object 3, then object 1 is the same length as object 3;

(*b*) if the length of object 1 is greater than the length of object 2 and object 2 is longer than object 3, then object 1 is longer than object 3; and

(*c*) if the length of object 1 is less than the length of object 2 and object 2 is shorter than object 3, then object 1 is shorter than object 3.

Being able to reason transitively is crucial for measurement and involves taking a stick, for instance, and using it as an instrument to judge whether two immovable towers are the same size. A child who can reason in this manner can take a third or middle item (the stick) as a referent by which to compare the heights or lengths of other objects.

Conservation of length is the understanding that as an object is moved, its length does not change.

To assess whether students at various ages could conserve length, one group of researchers showed them two strips of paper as illustrated in figure 1.1a. Most students agreed that the two strips were equal in length. However, when the interviewer moved the bottom strip forward a few centimeters, students who could not conserve length answered that the two strips were no longer equal (Piaget, Inhelder, and Szeminska 1960).

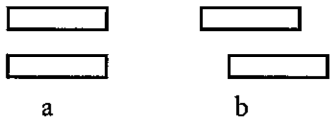

Fig. 1.1. Strips used in a conservation task

Over the past 40 years, researchers have debated both the ages and order in which conservation, transitivity, and measurement, in general, are developed. Many researchers agree that conservation is essential for, but not equivalent to, a full conception of measurement. Furthermore, Piaget, Inhelder, and Szeminska (1960) argued that transitivity is impossible for students who do not conserve lengths because once they move a unit, it is possible, in the student's view, for the length of the unit to change. Some researchers argue that students must reason transitively before they can understand measurement (Boulton-Lewis 1987; Kamii and Clark 1997). Therefore, these researchers go on to conclude that the ruler is useless as a measuring tool before a student can reason transitively (Kamii and Clark 1997). Although researchers agree that conservation is essential for a complete understanding of measurement, several articles caution that students do not necessarily need to develop transitivity and conservation before they can learn some measurement ideas. In fact, Clements (1999) argues that the only two that do seem to require conservation and transitivity are: (*a*) the inverse relation between the size of the unit and the number of those units and (*b*) the need to use equal length units when measuring. Although researchers do not agree on the order in which certain measurement ideas develop for students, we argue that children must develop each of these ideas to reach a full understanding of measurement regardless of the order of development.

The accumulation of distance means that the result of iterating a unit signifies, for students, the distance from the beginning of the first iteration to the end of the last. Furthermore, Piaget, Inhelder, and Szeminska (1960) characterized students' measuring activity as an accumulation of distance when the result of iterating forms nesting relationships to each other.

In Stephan et al. (2001), students measured the lengths of objects by pacing heel to toe and counting their steps. As one student paced the length of a rug, the teacher stopped her mid-measure and asked what she meant by "8." Some students claimed that 8 signified the space covered by the eighth foot, whereas others argued that it was the space covered from the beginning of the first foot to the end of the eighth. These latter students were measuring by accumulating distances. Piaget, Inhelder, and Szeminska (1960) contend that an accumulation of distance interpretation indicates that a student has constructed a complete understanding of linear measurement. Most researchers have observed this type of interpretation in nine- to ten-year-olds (Clements 1999; Kamii and Clark 1997; Piaget, Inhelder, and Szeminska 1960). However, Stephan et al. (2001) showed that, with meaningful instruction, children as young as six years old construct an accumulation of distance interpretation.

Relation between number and measurement—Measuring is related to number in that measuring is simply a case of counting. However, measuring is conceptually more advanced since students must reorganize their understanding of the very objects they're counting (discrete versus continuous units).

Many researchers have investigated the role that counting plays in students' development of measuring conceptions. Inhelder, Sinclair, and Bovet (1974) revealed that students make measurement judgments based on counting ideas. For example, they showed students two equal-length rows of matches (see fig. 1.2).

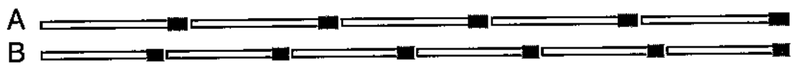

Fig. 1.2. Two rows of matches used in an interview

Although, from our perspective, the lengths of the rows were the same, many children argued that the row with six matches was longer because it had more matches. Other studies have also found that children draw on their counting experiences to interpret their measuring activity. Anyone who has taught measurement knows that students often start measuring with the numeral "1" as the starting point instead of 0. After all, when we measure, the first number word we say is "1." Lehrer (in press) argues that measurement assumes a "zero point," a point from which a measurement begins. The zero point need not be 0, but if students understand measuring only as "reading the ruler," then they will not understand this idea. Lubinski and Thiessan (1996) found that with meaningful instruction, students were able to use flexible starting points on a ruler to indicate measures successfully.

In summary, we have elaborated six important concepts that form the foundation for a full understanding of linear measurement: partitioning, unit iteration, transitivity, conservation, the accumulation of distance, and relation to number. Although researchers debate the order of the development of these concepts and the ages at which they are developed, they agree that these ideas form the foundation for measurement and should be considered during any measurement instruction. When a teacher has these ideas in mind during instruction, she is better able to ask questions that will lead them to construct these ideas. It is clear, however, that traditional measurement instruction is insufficient for helping students build these conceptions.

Instructional Activities That Build the Concepts

Traditionally, the goal of most measurement instruction has been to help students learn the skills necessary to use a conventional ruler. Although researchers agree that students ought to learn to use various measuring devices correctly, there has been a shift in instructional goals toward teaching students to develop the conceptual building blocks that lead to using rulers in meaningful ways. Such an emphasis has found its way into many new curricula in the form of starting measurement instruction with nonstandard units and moving towards the conventional ruler. However, there is some debate in recent literature as to the merits of starting instruction with only nonstandard units. We will not advocate one approach over another but rather emphasize that whichever instructional route is taken, the teacher should focus conversations and thoughts on the meaning that students' measuring activity has for them.

Nonstandard to Standard Devices

Most curricula advise a sequence of instruction in which students compare lengths, measure with nonstandard units, incorporate the use of manipulative standard units, and measure with a ruler (Clements 1999; Kamii and Clark 1997). The basis for this sequence is the developmental theory of measurement outlined by Piaget, Inhelder, and Szeminska (1960). Curriculum developers appear to assume that this approach motivates students to call for a standard measuring unit. Other researchers advocate this approach as well and argue that, when classroom discussions focus on students' meaning during measuring, they are able to construct sophisticated understanding (Lehrer in press; Lubinski and Thiessan 1996; Stephan et al. 2001).

Kamii and Clark (1997) stress that comparing lengths is at the heart of developing the notions of conservation, transitivity, and unit iteration, but most textbooks do not include these types of tasks. Textbooks tend to ask questions such as "How many paper clips does the pencil measure?" rather than "How much longer is the blue pencil than the red pencil?" Although Kamii and Clark advocate beginning instruction by comparing lengths with nonstandard or standard units (not a ruler), they caution that such an activity is often taught only as a procedure. Instead, teachers should focus students on the mental activity of transitive reasoning and accumulating distances. One type of task that involves indirect comparisons is to ask students if the doorway is wide enough for a table to go through. This involves an indirect comparison (and transitive reasoning) and therefore de-emphasizes physical measurement procedures.

Lehrer (in press) describes a sequence of instruction that a second-grade teacher, Ms. Clements, used. Her students began with pacing from one point to another. As students discussed their measuring activity, ideas concerning unit iteration and identical units emerged. Students progressed from counting paces to constructing a "footstrip" consisting of traces of their feet glued to a roll of adding-machine tape.

In a different classroom, the students wrestled with the idea of expressing their result in different-sized units (e.g., 15 paces or three footstrips, each of which had five paces). They also discussed how to deal with leftover space, to count it as a whole unit or as part of a unit. Measuring with these footstrips helped students think about length as a composition of these units. Furthermore, it provided the basis for constructing rulers (Stephan et al. 2001).

Beginning with Standard Measuring Devices

Other researchers encourage using an approach that is a bit different (Boulton-Lewis 1987; Clements 1999; Nunes, Light, and Mason 1993). They agree that very young children should have a variety of experiences comparing the size of objects in various dimensions; for example, finding all the objects in the room that are as long as their forearm. Later, however, rather than measuring initially only with nonstandard units, they argue that students benefit from using numerical measuring devices, either conventional rulers or nonstandard ones. In their study, Nunes, Light, and Mason (1993) found that children were successful on measurement tasks when they used a ruler, and this suggests that the numerical representation provided by rulers is not more difficult for children than starting with nonstandard units.

Clements (1999) and Boulton-Lewis (1987) suggest that using manipulative standard units or rulers is less demanding and more interesting for students. Clements (1999) suggests the following sequence of instruction. Students should be given a variety of experiences comparing the size of objects. Next, students should engage in experiences that allow them to connect number to length. Teachers should provide students with both conventional rulers and manipulatives units, such as unifix or centimeter cubes. As they explore with these tools the ideas of unit iteration (not leaving space between successive units, for example), correct alignment (with a ruler), and the zero-point concept can be developed. He cautions that teachers should focus on the meaning that the numerals on the ruler have for students, such as enumerating lengths rather than discrete numbers. He goes on to say that it is not until the second and third grade that teachers should introduce the need for standard units and the relation between the size and number of units.

Finally, children can construct measurement sense as they work in a computer microworld called GeoLogo (Clements 1999; Clements, Battista, and Sarama 1998). Clements posed problems asking students to estimate or calculate the length of lines and to draw lines of a given length with the Logo turtle. Some students just guessed without making or marking any units. Others created units they could count by drawing hashmarks, dots, or line segments to partition lengths. For some children, however, the segments they marked off were only equal if they were given units; these children's segments often corresponded to these units accurately. The most sophisticated students did not mark off units but drew proportional figures and visually partitioned line segments to assign them a length measure. According to Steffe (1991), young students can impose such a "conceptual ruler" onto objects and geometric figures.

In a related study, Barrett and Clements (2000) found that introducing perimeter tasks not only teaches that important concept, but also introduces children to the need for coordinating measures of parts of paths with the measure around the entire path. Perimeter tasks also emphasize measurable attributes of units as children examine grids and other ways of partitioning the sides and perimeter of a shape. By setting tasks that require a child to identify measured features, like focusing on the edges of a square tile rather than the entire tile as a unit, children learn to discriminate length from area.

AREA MEASUREMENT

Area is an amount of two-dimensional surface that is contained within a boundary and that can be quantified in some manner (Baturo and Nason 1996). Reynolds and Wheatley (1996) explain that the measure of a region is determined by comparing the region to another smaller unit, usually a square unit. They go on to say that there are at least four assumptions involved when assigning a number to a region: (1) a suitable two-dimensional region is chosen as a unit, (2) congruent regions have equal areas, (3) regions do not overlap, and (4) the area of the union of two regions is the sum of their areas. Therefore, finding the area of a region can be thought of as tiling (partitioning) a region with a two-dimensional unit of measure.

Many difficulties abound for students as they learn to measure area. First, the formal method for figuring rectangular areas is to measure the lengths of two sides and then multiply these one-dimensional units to construct a two-dimensional measure. Such a mental activity is sophisticated even for intermediate and middle-school students. Second, the result of iterating a unit along a rectangular region creates an array of units. Although it is obvious to adults that an array is created, the structure of an array is conceptually complex for students. Third, interpreting the area formula for rectangular regions requires reasoning multiplicatively about the product of two lengths. Research has found that reasoning multiplicatively with respect to area is not trivial (Simon and Blume 1994). Finally, often the tools and procedures used in measuring area mask the intended conceptual aspects that underlie area measurement.

Important Concepts in Area Measurement

There are at least four foundational concepts that are involved in learning to measure area: (1) partitioning, (2) unit iteration, (3) conservation, and

(4) structuring an array. As with linear measurement, *partitioning* is the mental act of cutting two-dimensional space with a two-dimensional unit. Teachers often assume that the product of two lengths structures a region into an area of two-dimensional units for students. However, the literature suggests that the construction of a two-dimensional array from linear units is nontrivial. Lehrer (in press) explains that students' first experiences with area might include tiling a region with a two-dimensional unit of choice and, in the process, discuss issues of leftover spaces, overlapping units, and precision to name a few. Discussions of these ideas lead students to partition a region mentally into subregions that can be counted.

Unit iteration is another important concept that students construct as they cover regions with area units. There should be no gaps or overlapping of units. Students also tend to fill in the region with units, but do not extend units over the boundaries of the bigger region (cf., Stephan et al. 2001). Furthermore, when students are given a choice, they choose units that physically resemble the region they are covering. For instance, Nunes, Light, and Mason (1993) found that children chose brick manipulatives to cover a rectangular region, whereas Lehrer's (in press) students used beans to cover an outline of their hands. However, they also reported that students had no problem mixing rectangular and triangular shapes to cover the same region. Finally, the literature stresses that the result of iterating area units ought to signify an array structure for students. Although the structure of an array is conceptually difficult, Outhred and Mitchelmore (2000) suggest that second-grade students are capable of constructing this relationship.

Similar to linear measurement, *conservation* of area is an important idea that is often neglected in instruction. Students have difficulty accepting that when they cut a given region and rearrange its parts to form another shape, the area remains the same (Lehrer, in press). Students should explore and discuss the consequences of folding or rearranging pieces to establish that one region, cut and reassembled, covers the same space. Related research shows that young children use different strategies to make judgments of area. For example, four- and five-year-olds use height + width rules to make area judgments (Cuneo 1980). Children from six to eight years use a linear extent rule, such as the diagonal of a rectangle. Only after this age do most children move to multiplicative rules. This leads to our next concept.

As we have stressed earlier, *structuring an array* is an extremely sophisticated process for students, particularly in the early grades. Battista et al. (1998) argue that students must learn such structuring to understand area. They report that children develop through a series of levels in this learning.

- **Level 1:** No use of a row or column of squares as a composite unit (a "line" of squares thought of as a group). Students at this level have difficulty visualizing the location of squares in an array and counting square tiles that cover the interior of a rectangle.

- **Level 2:** Partial row or column structuring. Some students, for example, make two rows but no more.
- **Level 3A:** Structuring an array as a set of row- or column-composites. Students at this level see the rectangle as covered by copies of composite units (rows or columns) but cannot coordinate those with the other dimension.
- **Level 3B:** Visual row- or column-iteration. These students can iterate a row (e.g., count by fours) if they can see those rows.
- **Level 3C:** Interiorized row- or column-iteration. These students can iterate a row using the number of squares in a column. Only at this level is the usual "formula" method of determining area going to have a firm conceptual basis for most students.

One way to encourage students to construct arrays of units is to have them tile a rectangular region and keep count. However, Outhred and Mitchelmore (2000) caution that using wooden or plastic tiles is too easy a task. These manipulatives mask the structure of an array because students cannot overlap the tiles, for instance. Instead, as students create arrays (with tiles), they should also be encouraged to draw the results of their covering (cf., Reynolds and Wheatley 1996). Drawing the tiles that cover a region leads to surprising pictures showing that the array structure was not as apparent to children as to adults. For example, some students draw a series of square tiles within the region they were measuring, yet there are gaps between tiles. Other students draw arrays that have unequal number of units in each row. Students need to be provided tasks and instruction that leads them through the levels of learning this structuring (Akers et al. 1997; Battista et al. 1998).

Instructional Approaches

What kind of activities help students learn initial area concepts, structure arrays, and finally learn all five concepts to form a complete foundation for measuring area meaningfully? First, students should investigate covering regions with a unit of measure. They should realize that there are to be no gaps or overlapping and that the entire region should be covered. Second, they should learn how to structure arrays. This is a long-term process that can be started in primary grades. Figuring out how many squares in pictures of arrays, with less and less graphic information of clues, is an excellent task (see Akers et al. 1997; Battista et al. 1998). Third, students should learn that the length of the sides of a rectangle can determine the number of units in each row and the number of rows in the array. Fourth, often in the intermediate grades, students who can structure an array can meaningfully learn to multiply the two dimensions as a shortcut for determining the total number of squares. However, if students do not understand array structures, they will have difficulty fully understanding multiplicative formulas for area.

This sequence of conceptual development is similar to the instructional approach suggested by Lehrer, Jacobson, et al. (1998). They suggest that students' development should proceed from informal measurement to more formal procedures. They, along with Nunes, Light, and Mason (1993), caution that teachers should not begin area instruction with rulers. Nunes, Light, and Mason report that students in their study failed to solve area problems when they used a ruler but were able to devise multiplicative solutions when given a chance to cover with a unit. If instruction begins with a ruler, one of the most common mistakes is for children to measure the length of each side and add the two linear measures together. Therefore, Lehrer, Jacobson, et al. (1998) suggests engaging students in tasks requiring them to find the area of an irregular surface with a unit of their choice. For example, their students were asked to trace their hands and find their area using a variety of manipulatives (e.g., centimeter cubes, beans). Although most children chose objects that physically resemble the shape of their hands (i.e., beans), this task provided the opportunity to discuss how to deal with leftover space that was uncovered. Because the students were unsure how to solve this problem, the teacher introduced a square grid as a measurement device. They gradually accepted this notation and used it to estimate and combine partial units.

As a follow-up task students can be asked to draw and measure islands with their newly constructed square grids. This type of task gives students more opportunity to measure with square units and to combine parts of units together to form whole units. It is important to note, say Lehrer, Jacobson, et al., that the teacher must not focus on the calculational processes students develop but rather on the meaning that their procedures have for them. Students may be moved towards building arrays with tasks such as finding the area of zoo cages. Lehrer's students were given a set of various polygonal outlines that represented the floor plan of different zoo cages. Students were provided with rulers if they found them necessary. Although some students measured the lengths of each side of a rectangle, they incorrectly argued that the resulting area would be 40 inches. Other students partitioned the rectangular cages into array structures and argued that they really meant 40 square units. In this way, students were provided a chance to relate the familiar array structure to ideas of length. The reader is invited to see the following references for more detail concerning these types of tasks (Akers et al. 1997; Lehrer, Jenkins, and Osana 1998; Lehrer in press).

In summary, the too-frequent practice of simple counting of units to find area (achievable by preschoolers) leading directly to teaching formulas underemphasizes the conceptual basis of area measurement. Instead, educators should build on young children's initial spatial intuitions and appreciate the need for students to (*a*) construct the idea of measurement units (including measurement sense for standard units); (*b*) have many experiences covering quantities with appropriate measurement units and counting those units; (*c*) structure spatially the object they are to measure; and (*d*) construct the inverse relationship between the size of a unit and the number of units used in a measurement.

CONCLUSION

We began this paper by claiming that students do not develop sophisticated understandings of measurement. We believe the reason that students' understanding of measurement lags behind most other mathematical ideas is that measurement instruction tends to focus on learning the *procedures* for measuring rather than *big ideas* about linear and two-dimensional space. The research indicated that developing measurement sense is more complex than learning the skills or procedures for determining a measure. Rather, researchers emphasize the importance of partitioning space into equal-sized units (either linear units or arrays of two-dimensional units) and counting the iterations of that unit. In order to develop these big ideas teachers must create classroom environments in which students engage in multiple measuring situations that encourage students to measure with standard and nonstandard units. Important to all measurement instruction, classroom conversations should be saturated with talk about the meaning that students' measuring has for them, not merely students' explanations of their procedures.

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